(Non)singular Kantowski-Sachs universe from quantum spherically reduced matter

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Using the *s*-wave and large-*N* approximation the one-loop effective action for 2D dilaton coupled scalars and spinors which are obtained by spherical reduction of 4D minimal matter is found. Quantum effective equations for reduced Einstein gravity are written. Their analytical solutions corresponding to a 4D Kantowski-Sachs (KS) universe are presented. For quantum-corrected Einstein gravity we get a nonsingular KS cosmology which represents (1) a quantum-corrected KS cosmology which existed on a classical level or (2) a purely quantum solution which had no classical limit. The analogy with the Nariai BH is briefly mentioned. For purely induced gravity (no Einstein term) we found a general analytical solution but all KS cosmologies under discussion are singular. The corresponding equations of motion are reformulated as a classical mechanics problem of motion of a unit mass particle in some potential *V*. [S0556-2821(99)08112-6]

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I. INTRODUCTION

It is a quite common belief that two-dimensional dilatonic gravity may be useful only as a toy model for the study of realistic 4D gravity, especially in the quantum regime (for a review of quantum gravity, see, for example, Ref. [1]). However, it is quite well known (for example, see Ref. [2]) that spherical reduction of Einstein gravity leads to some specific dilatonic gravity (for its most general model, see Ref. [3]). At the same time, the spherical reduction of minimal 4D matter leads to 2D dilaton coupled matter.

The conformal anomaly for a 2D conformally invariant, dilaton-coupled scalar has been found in Refs. [4–7] and the corresponding anomaly-induced effective action has been found in Refs. [5,6,8,9]. The same calculation for 2D and 4D dilaton-coupled spinors has been presented recently in [10]. Using such an anomaly-induced effective action (i.e., working in the *s*-wave and large-*N* approximation) and adding it to the reduced Einstein action one may study four-dimensional Kantowski-Sachs (KS) quantum cosmology [11] in a consistent way as it was done in Ref. [12] (for a discussion of 2D dilatonic quantum cosmology, see, for example, [13–15,12]). (Note that using similar methods the inducing of wormholes in the early Universe has been recently investigated in Ref. [16], confirming such inducing.)

In the (mainly numerical) study of Refs. [12] it was found that most of the KS cosmologies under investigation are singular at the initial stage of the evolution of the Universe. The interesting question is, can we construct (non)singular KS quantum cosmologies using purely analytical methods?

In the present work we try to answer this question. Using an analogy between the KS cosmology and Schwarzschild black hole (BH) (or its generalizations) after the interchange of time and radial coordinates (see [11,17]) we found a particular solution of the quantum equations of motion analytically. This solution represents nonsingular KS cosmology (expanding Universe with always a nonzero radius) which comes from Schwarzschild–de Sitter (or –anti-de Sitter) BH after interchange of time and radial coordinates. For purely induced gravity (when the cosmology is defined completely by the quantum effects of matter) we present a general analytical solution of the quantum equations of motion. Unfortunately, in this case all found KS quantum cosmologies are singular. We also reformulate the last problem as a classical mechanics problem, rewriting the system of equations as describing the motion of unit mass particles in some potential *V*.

II. ANOMALY-INDUCED EFFECTIVE ACTION AND NONSINGULAR KS COSMOLOGY

We will start from the action of Einstein gravity with N minimal real scalars and M Majorana fermions:

$$S=-\frac{1}{16\pi G}\int d^4x\sqrt{-g_{(4)}}(R^{(4)}-2\Lambda)+\int d^4x\sqrt{-g_{(4)}}\left(\frac{1}{2}\sum_{i=1}^{N} g_{(4)}^{\alpha\beta}\partial_\alpha\chi_i\partial_\beta\chi_i+\sum_{i=1}^{M}\bar{\psi}_i\gamma^\mu\nabla_\mu\psi_i\right), \tag{1}$$

where χ_i and ψ_i are real scalars and Majorana spinors, respectively. In order to apply the large-N approach N and M are considered to be large, $N,M\gg 1$; G and Λ are gravitational and cosmological constants. Note that we do not discuss possible topological restrictions to spinors (such as

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twisted spinors, etc.) as it does not matter in our discussion; the anomaly-induced effective action will be the same.

We now assume spherically symmetric spacetime:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + e^{-2\phi}d\Omega^2, \tag{2}$$

where $\mu,\nu=0,1$, $g_{\mu\nu}$ and ϕ depend only from $x^0=t$, and $d\Omega^2$ corresponds to a two-dimensional sphere.

The action (1) reduced according to Eq. (2) takes the form

$$S_{red} = \int d^2x\sqrt{-g}\,\mathrm{e}^{-2\phi}\left[-\frac{1}{16\pi G}\{R+2(\nabla\phi)^2 -2\Lambda+2\mathrm{e}^{2\phi}\} + \frac{1}{2}\sum_{i=1}^{N}(\nabla\chi_i)^2 + \sum_{i=1}^{2M}\bar{\psi}\gamma^\mu\nabla_\mu\psi\right]. \tag{3}$$

Note that the fermion degrees of freedom after reduction are twice the original ones. In the spherical reduction $\gamma^\mu\nabla_\mu$ is repalaced by $\gamma^\mu(\nabla_\mu+\partial_\mu^\phi)$ but the second term does not contribute to the action.

Working in the large-N and s-wave approximation one can calculate the quantum correction to S_{red} (effective action). Using a 2D conformal anomaly for the dilaton-coupled scalar and dilaton-coupled spinor, one can find the anomaly-induced effective action. In the case of dilaton absence such induced effective action gives the complete effective action which is valid for an arbitrary two-dimensional background. In the presence of a dilaton as above the complete effective action consists of two pieces. The first one is induced effective action which is given actually for any background but with an accuracy up to the conformally invariant functional. The second piece of it, i.e., the conformally invariant functional, cannot be found in closed form in the case of scalars. Nevertheless, one can find it, using standard methods as some expansion, say, on the curvatures. We use the Schwinger-DeWitt (SD) technique to calculate it. We keep only the leading part of such an expansion; for more details and an explanation, see Refs. [6,8,10]. Then the effective action may be written in the following form [6,8,10]:

$$W = -\frac{1}{8\pi}\int d^2x\sqrt{-g}\left[\frac{N+M}{12}R\frac{1}{\Delta}R - N\nabla^\lambda\phi\nabla_\lambda\phi\frac{1}{\Delta}R + \left(N+\frac{2M}{3}\right)\phi R + 2N\ln\mu^2\nabla^\lambda\phi\nabla_\lambda\phi\right]. \tag{4}$$

Note that the numerical coefficient in front of the logarithmic term does not matter as it can be changed by a rescaling of μ. As was shown in Ref. [10] dilaton coupled spinors do not give a contribution to this term, at least in leading order of the SD expansion. The anomaly-induced effective action for dilaton-coupled spinors is found also in the Appendix following Ref. [10].

The equations of motion may be obtained by variation of $\Gamma=S_{red}+W$ with respect to $g^{\pm\pm}$, $g^{\pm\mp}$, and ϕ:

$$0 = \frac{\mathrm{e}^{-2\phi}}{4G}[2\partial_t\rho\partial_t\phi+(\partial_t\phi)^2-\partial_t^2\phi] - \frac{N+M}{12}[\partial_t^2\rho-(\partial_t\rho)^2] - \frac{N}{2}\left(\rho+\frac{1}{2}\right)(\partial_t\phi)^2 - \frac{N+2M/3}{4}(2\partial_t\rho\partial_t\phi-\partial_t^2\phi) - \frac{N}{4}\ln\mu^2(\partial_t\phi)^2 + t_0, \tag{5}$$

$$0 = \frac{\mathrm{e}^{-2\phi}}{8G}[2\partial_t^2\phi-4(\partial_t\phi)^2+2\Lambda\mathrm{e}^{2\rho}-2\mathrm{e}^{2\rho+2\phi}] + \frac{N+M}{12}\partial_t^2\rho + \frac{N}{4}(\partial_t\phi)^2 - \frac{N+2M/3}{4}\partial_t^2\phi, \tag{6}$$

$$0 = -\frac{\mathrm{e}^{-2\phi}}{4G}[-\partial_t^2\phi+(\partial_t\phi)^2+\partial_t^2\rho-\Lambda\mathrm{e}^{2\rho}] + \frac{N}{2}\partial_t(\rho\partial_t\phi) + \frac{N+2M/3}{4}\partial_t^2\rho + \frac{N}{2}\ln\mu^2\partial_t^2\phi. \tag{7}$$

Here we have chosen the conformal gauge $g_{\pm\mp}=-\frac{1}{2}\mathrm{e}^{2\rho}$, $g_{\pm\pm}=0$ ($x^\pm\equiv t\pm r$), and t_0 is a constant which is determined by the initial conditions. Combining Eqs. (5) and (6) we get

$$0 = \frac{\mathrm{e}^{-2\phi}}{4G}[-(\partial_t\phi)^2+2\partial_t\rho\partial_t\phi+\Lambda\mathrm{e}^{2\rho}-\mathrm{e}^{2\rho+2\phi}] + \frac{N+M}{12}(\partial_t\rho)^2 - \frac{N}{2}\rho(\partial_t\phi)^2 - \frac{N+2M/3}{2}\partial_t\rho\partial_t\phi - \frac{N}{4}\ln\mu^2(\partial_t\phi)^2 + t_0. \tag{8}$$

This equation may be used to determine t_0 from the initial condition; it decouples from the remaining two equations. Hence, Eq. (8) is not necessary in the subsequent analysis.

It is often convienient to use the cosmological time τ instead of t, where the metric is given by

$$ds^2 = -d\tau^2 + \mathrm{e}^{2\rho}dr^2 + \mathrm{e}^{-2\phi}d\Omega^2. \tag{9}$$

Since we have $d\tau=\mathrm{e}^\rho dt$, we obtain $\partial_t=\mathrm{e}^\rho\partial_\tau$ and $\partial_t^2 = \mathrm{e}^{2\rho}(\partial_\tau^2+\partial_\tau\rho\partial_\tau)$. Then Eqs. (6) and (7) may be rewritten as follows:

$$0 = \left(\frac{\mathrm{e}^{-2\phi}}{G}-N-\frac{2M}{3}\right)\partial_\tau^2\phi + \frac{N+M}{3}\partial_\tau^2\rho + \left(-\frac{2\mathrm{e}^{-2\phi}}{G}+N\right)(\partial_\tau\phi)^2 + \left(\frac{\mathrm{e}^{-2\phi}}{G}-N-\frac{2M}{3}\right)\partial_\tau\rho\partial_\tau\phi - \frac{1}{G}(-\Lambda\mathrm{e}^{-2\phi}+1) + \frac{N+M}{3}(\partial_\tau\rho)^2, \tag{10}$$

$$0=-\left(\frac{e^{-2\phi}}{G}-N-\frac{2M}{3}\right)\partial_\tau^2\rho+\left\{\frac{e^{-2\phi}}{G}+2N(\rho+\ln\mu^2)\right\}\times\partial_\tau^2\phi-\frac{e^{-2\phi}}{G}(\partial_\tau\phi)^2+\left\{\frac{e^{-2\phi}}{G}+2N(\rho+1+\ln\mu^2)\right\}\times\partial_\tau\rho\,\partial_\tau\phi-\left(\frac{e^{-2\phi}}{G}-N-\frac{2M}{3}\right)(\partial_\tau\rho)^2+\frac{e^{-2\phi}}{G}\Lambda. \quad (11)$$

We now consider a special solution for Eqs. (5), (6) and (7) corresponding to the (Wick-rotated) Nariai solution [18], where ϕ is a constant: $\phi=\phi_0$. Then Eqs. (6) and (7) can be rewritten as follows:

$$0=\frac{3}{(N+M)G}(\Lambda e^{-2\phi_0}-1)e^{2\rho}+\partial_t^2\rho, \quad (12)$$

$$0=\frac{\Lambda e^{-2\phi_0}}{G}\left(-\frac{e^{-2\phi_0}}{G}+N+\frac{2}{3}M\right)^{-1}e^{2\rho}+\partial_t^2\rho. \quad (13)$$

Comparing Eq. (12) with Eq. (13), we obtain

$$e^{-2\phi_0}=\frac{(2N+M)G}{6}+\frac{1}{2\Lambda}\pm\frac{1}{2}\sqrt{\frac{(2N+M)^2G^2}{9}+\frac{1}{\Lambda^2}-\frac{(8N+6M)G}{3\Lambda}}. \quad (14)$$

The sign $\pm$ in Eq. (14) should be $+$ if we require that the solution coincide with the classical one $e^{-2\phi_0}=1/\Lambda$ in the classical limit of $N,M\to 0$. On the other hand, in the solution with the $-$ sign, we have $e^{-2\phi_0}\sim(3N+2M)G/3\to 0$ in the classical limit. Therefore the second solution does not correspond to any classical solution but the solution is generated by the quantum effects.

The solution of Eqs. (12) and (13) is given by

$$e^{2\rho}=\begin{cases}\dfrac{2C}{R_0}\dfrac{1}{\cos^2(t\sqrt{C})} & \text{when } R_0>0,\\ -\dfrac{2C}{R_0}\dfrac{1}{\cosh^2(t\sqrt{C})} & \text{when } R_0<0.\end{cases} \quad (15)$$

Here $C>0$ is a constant of the integration and R_0 is the 2D scalar curvature, which is given by

$$R_0=2e^{-2\rho}\partial_t^2\rho=-\frac{3\Lambda}{(N+M)G}\left(\frac{(2N+M)G}{3}-\frac{1}{\Lambda}\pm\sqrt{\frac{(2N+M)^2G^2}{9}+\frac{1}{\Lambda^2}-\frac{(8N+6M)G}{3\Lambda}}\right). \quad (16)$$

Note that the 4D curvature $R_4=R_0+2e^{2\phi_0}$ becomes a constant. It should be also noted that the solution exists for both cases of positive Λ and negative Λ. In Eq. (16), the $+$ sign corresponds to the classical limit (N, $M\to 0$). In the limit, we obtain $R_0\to 2\Lambda$ ($R_4\to 4\Lambda$). On the other hand, the $-$ sign in Eq. (16) corresponds to the solution with the $-$ sign in Eq. (14) generated by the quantum effect. In the classical limit for the solution, the curvature R_0 in Eq. (16) diverges as $R_0\sim 3/2(N+M)G\to+\infty$. Therefore, from Eq. (15), we find that $e^{2\rho}$ vanishes (note that $R_0>0$ in the limit): $e^{2\rho}=4(N+M)GC/3\cos^2(t\sqrt{C})\to 0$. Therefore, by using Eq. (2), we obtain the following metric near the classical limit:

$$ds^2=\frac{4(N+M)GC}{3\cos^2(t\sqrt{C})}(-dt^2+dr^2)+\frac{(3N+2M)G}{3}d\Omega^2. \quad (17)$$

This is a nonsingular metric for fixed N, M.

It should be interesting to consider the limit $\Lambda\to 0$, where there is no de Sitter or anti–de Sitter (AdS) solution at the classical level. In the limit, we can have a finite solution

$$e^{-2\phi_0}\to\frac{(3N+2M)G}{3},$$

$$e^{2\rho}\to\frac{(N+M)GC}{3\cos^2(t\sqrt{C})}\left(R_0\to\frac{6}{(N+M)G}\right). \quad (18)$$

This tells us that the Nariai space can be generated by the quantum effect even if $\Lambda=0$.

The obtained solution (15) [and (17)] might appear to have a singularity when $\cos^2(t\sqrt{C})=0$ (for the $R_0>0$ case) but the singularity is an apparent one. In fact the scalar curvature R_0 in Eq. (16) is always constant. If we change the conformal time coordinate t by the cosmological time τ in Eq. (9), we find that the time $\cos^2(t\sqrt{C})=0$ corresponds to infinite future or past.

In the following, we assume $R_0>0$ for simplicity. The $R_0<0$ case can be easily obtained by changing the constant $C\to -C$ and analytically continuing solutions. We now change the time coordinate by

$$\tau=\sqrt{\frac{2}{R_0}}\ln\left(\frac{1+\tan(t\sqrt{C}/2)}{1-\tan(t\sqrt{C}/2)}\right).$$

Then the time $\cos^2(t\sqrt{C})=0$ ($t\sqrt{C}=\pm\pi/2$) corresponds to $\tau=\pm\infty$. Using the cosmological time τ, we obtain the following metric:

$$ds^2=-d\tau^2+\frac{2C}{R_0}\cosh^2\left(\tau\sqrt{\frac{R_0}{2}}\right)dr^2+e^{-2\phi_0}d\Omega^2. \quad (19)$$

Here $e^{-2\phi_0}$ is given in Eq. (14). If we assume r has periodicity of 2π, the metric describes a nonsingular Kantowski-Sachs universe, whose topology is $S_1\times S_2$. The radius of S_2 is constant but the radius of S_1 has a minimum when $\tau=0$ and increases exponentially with the absolute value of τ.

Hence we found a nonsingular KS cosmology which exists on the classical level and which also exists on the quantum level (as a quantum-corrected KS cosmology). This metric may be considered as the one obtained from the

Schwarzschild–de Sitter (Nariai) BH (for positive cosmological constant) [18] and from the Schwarzschild–anti-de Sitter BH (for negative cosmological constant). To make the correspondence one has to interchange time and radial coordinates assuming a corresponding Wick rotation. It is very interesting that the last case (of negative cosmological constant) may be relevant to AdS conformal field theory (CFT) [19] correspondence. We also found a nonsingular KS universe which does not have the classical limit and which is completely induced by quantum effects (even in the case of zero cosmological constant). Hence we obtained an expanding universe with radius which is never zero. This cosmology may be interesting in the framework of inflationary universes as it can describe some substage of inflationary universes where there is an effective expansion only along one (or two) space coordinates.

III. INDUCED GRAVITY AND SINGULAR KS QUANTUM COSMOLOGY

Let us discuss now the situation when we live in the regime where quantum (nonlocal) anomaly-induced effective action gives a major contribution to equations of motion. In other words, quantum cosmology is defined completely by quantum effects (effective gravity theory which at some point makes the transition to classical gravity). As we will see in this case the equations of motion admit analytical solutions which lead to a singular KS cosmology.

We consider purely induced gravity, i.e., the $N,M\to\infty$ case. Then the Einstein action can be dropped. For this case, the field equations have the form

$$0=-\left(N+\frac{2M}{3}\right)\partial_\tau^2\phi+\frac{N+M}{3}\partial_\tau^2\rho+N(\partial_\tau\phi)^2 -\left(N+\frac{2M}{3}\right)\partial_\tau\rho\partial_\tau\phi+\frac{N+M}{3}(\partial_\tau\rho)^2, \quad (20)$$

$$0=\left(N+\frac{2M}{3}\right)\partial_\tau^2\rho+2N(\rho+a)\partial_\tau^2\phi +2N(\rho+1+a)\partial_\tau\rho\partial_\tau\phi+\left(N+\frac{2M}{3}\right)(\partial_\tau\rho)^2, \quad (21)$$

where $a=\ln\mu^2$.

Equation (21) admits the following integral of motion:

$$I_1=e^{\rho}\left[\left(N+\frac{2}{3}M\right)\rho'+2N(\rho+a)\phi'\right]. \quad (22)$$

Here $'=d/d\tau$. For the case $\phi=\text{const}=\phi_0$, we have the following solution of Eqs. (20),(21):

$$r(\tau)=\text{const}=e^{-\phi_0}, \quad f(\tau)=e^{\rho}=(f_0'\tau+f_0). \quad (23)$$

For the metric (9), the scalar curvature has the following form:

$$R=2(e^{2\phi}+3\phi'^2-2\phi'\rho'+\rho'^2-2\phi''+\rho''). \quad (24)$$

Then for the solution (23) we have $R=2e^{2\phi_0}=\text{const}$. For the case $\rho=\text{const}=\rho_0$, we have the solution of Eqs. (20) and (21):

$$\rho_0=-a, \quad r(\tau)=e^{-\phi}=(c_1\tau+c_2)^{1+2M/3N}, \quad c_1,c_2=\text{const}, \quad (25)$$

$$c_2=\exp\left(\frac{-3N\phi_0}{2M+3N}\right), \quad c_1=c_2\frac{-3N\phi_0'}{(2M+3N)}. \quad (26)$$

Here ϕ_0 and ϕ_0' are the values of ϕ and ϕ' at $\tau=0$, respectively. For the solution (25) we have the following scalar curvature:

$$R=\frac{2\left(c_1^2(4M^2+8MN+3N^2)+\frac{3N^2}{(c_2+c_1\tau)^{4M/3N}}\right)}{3N^2(c_2+c_1\tau)^2}. \quad (27)$$

The solution (25) has a singularity at $\tau=-c_2/c_1$.

If $\rho\neq\text{const}$, then we obtain the following special solution:

$$f(\tau)=e^{\rho}=f_0'\tau+f_0,$$

$$r(\tau)=e^{-\phi}=c_3[a+\ln(f_0'\tau+f_0)]^{1+2M/3N}. \quad (28)$$

Here f_0 and f_0' are the values of $f=e^{\rho}$ and f' at $\tau=0$, respectively. For the solution (28) we have

$$R=\frac{2\left(\frac{f_0'^2(4M^2+8MN+3N^2)}{N^2(f_0'\tau+f_0)^2}+\frac{3e^{2\phi_0}(a+\rho_0)^{2+4M/3N}}{(a+\ln(f_0'\tau+f_0))^{4M/3N}}\right)}{3[a+\ln(f_0'\tau+f_0)]^2}. \quad (29)$$

The solution (28) has a singularity when $\tau=-f_0/f_0' - e^{-a}/f_0$.

Let us consider the case when $I_1=0$, $\rho\neq$const; then we have the following special solution:

$$\frac{d\tilde{\rho}}{d\tau}=\pm\frac{6N}{2M+3N}\tilde{\rho}e^{-\tilde{\rho}}\sqrt{\frac{c_1}{\tilde{\rho}\left[1+\frac{6(M+N)}{(2M+3N)^2}\tilde{\rho}\right]}},$$

$$\tilde{\rho}=\rho+a, \tag{30}$$

$$r(\tau)=e^{-\phi}=c_2|\tilde{\rho}|^{\pm(2M+3N)/6N}, \quad c_1,c_2=\text{const}. \tag{31}$$

For the solution (30),(31), we have the following scalar curvature:

$$R=e^{2\phi}+\frac{3c_1(2M+3N)}{e^{2\tilde{\rho}}\tilde{\rho}[(2M+3N)^2+6(M+N)\tilde{\rho}]^2}\times\{(2M+N)(2M+3N)^2+6[2M^2+MN(2N+1)+N^2(3N-1)]\tilde{\rho}\}. \tag{32}$$

The solution (30),(31) is singular when $\tilde{\rho}=0$ or $\tilde{\rho}=-(2M+3N)^2/6(M+N)$.

We now consider more general cases. First we should note that Eqs. (20) and (21) admit one more integral besides I_1 in Eq. (22):

$$I_2=e^{2\rho}\left[\frac{N+M}{12}(\rho')^2-\frac{N}{2}\rho(\phi')^2-\frac{N+\frac{2}{3}M}{2}\rho'\phi'-\frac{N}{2}a(\phi')^2\right]. \tag{33}$$

Since Eq. (22) can be solved with respect to ϕ',

$$\phi'=\frac{1}{2N(\rho+a)}\left[-\left(N+\frac{2}{3}M\right)\rho'+I_1e^{-\rho}\right], \tag{34}$$

we can delete ϕ by substituting Eq. (34) into Eq. (33) and we obtain

$$0=\frac{1}{2}(\rho')^2+V(\rho), \quad V(\rho)=-\frac{6I_2}{N+M}\frac{\rho+a-\alpha}{\rho+a-\beta}e^{-2\rho}. \tag{35}$$

Here $\alpha\equiv -I_1^2/8NI_2$ and $\beta\equiv-3(N+\frac{2}{3}M)^2/2N(N+M)$. Note that β is negative and α is positive (negative) when I_2 is negative (positive).

Since the 4D scalar curvature is given in Eq. (24), Eq. (35) tells that there would be a curvature singularity when $\rho+a\to\beta\pm0$. In fact, when $\rho+a\sim\beta\pm0$, we obtain, from Eq. (35), $(\rho')^2\sim A/(\rho+a-\beta)$ $[A\equiv 12I_2(\beta-\alpha)e^{2(a-\beta)}/(N+M)]$. Therefore we find

$$\rho+a=\beta+\left(\frac{3A}{2}(\tau-\tau_\beta)\right)^{2/3}. \tag{36}$$

Here τ_β is a constant of the integration and $\rho+a=\beta$ when $\tau=\tau_\beta$. By substituting Eq. (36) into Eq. (24), we find the behavior of the scalar curvature R:

$$R\sim\frac{4N}{27\left(N+\frac{2}{3}\right)}\left(\frac{3A}{2}\right)^{2/3}(\tau-\tau_\beta)^{-4/3}+\frac{4M^2}{243\left(N+\frac{2}{3}M\right)^2}\left(\frac{3A}{2}\right)^{2/3}(\tau-\tau_\beta)^{-2/3}. \tag{37}$$

Therefore there is always a singularity when $\rho+a\sim\beta\pm0$ except the $\alpha=\beta$ case, when A vanishes (we should note that A is finite when $I_2=0$).

In the case $\alpha=\beta$, Eq. (35) can be explicitly solved to give

$$e^{\rho}=\pm\frac{12I_2}{N+M}(\tau-\tau_0). \tag{38}$$

Here τ_0 is a constant of the integration. Equation (38) tells that there is a singularity when $\tau=\tau_0$. In the case of an expanding universe [+ sign in Eq. (38)], Eq. (34) tells us that $\phi'=0$; i.e., ϕ is a constant as in the Nariai space [18] (note that there is a singularity even in this case, which is different from the Nariai space). On the other hand, in the case of a shrinking universe [− sign in Eq. (38)], from Eq. (34), we obtain

$$\phi'=-\frac{(N+M)I_1}{2NI_2(\tau_0-\tau)\left\{a+\ln\left(-\frac{12I_2(\tau-\tau_0)}{N+M}\right)\right\}}. \tag{39}$$

Equation (39) tells that there is a curvature singularity when $\tau-\tau_0=[-(N+M)/12I_2]e^{-a}$ besides $\tau=\tau_0$.

Equation (35) might tell that there would be a kind of singularity (not always a curvature singularity) when $\rho+a\to\alpha\pm0$. We now investigate the behavior near $\rho+a\to\alpha\pm0$. Then Eq. (35) has the form of $(\rho')^2\sim B(\rho+a-\alpha)$ $[B\equiv 12I_2e^{2(a-\alpha)}/(N+M)(\alpha-\beta)]$. B should be positive (negative) if $\rho+a\to\alpha+0$ ($\rho+a\to\alpha-0$) since $(\rho')^2\geqslant0$. Then we obtain

$$\rho+a\sim\alpha+\frac{B}{4}(\tau-\tau_\alpha)^2. \tag{40}$$

Here τ_α is a constant of integration and $\rho+a=\alpha$ when $\tau=\tau_\alpha$. Equation (40) tells that ρ is ''reflected'' (i.e., ρ' changes its sign) at $\tau=\tau_\alpha$ smoothly without curvature singularity.

Equation (34) also tells that there might be a singularity when $\rho+a=0$. When $\rho+a\sim0$, the behavior of ρ' is given from Eq. (35) by

$$\rho' \sim \pm \frac{I_1 e^a}{N+\frac{2}{3}M}\left\{1-\frac{1}{2}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)(\rho+a)\right\}. \tag{41}$$

Substituting Eq. (41) into Eq. (35), we find that there is no singularity at $\rho+a=0$ if the $+$ sign in Eq. (41) is chosen. This means that the singularity does not appear $\rho'>0$ if $I_1>0$ or $\rho'<0$ if $I_1<0$ but the singularity would appear in other cases since Eq. (35) tells us that

$$\phi-\phi_0 \sim \frac{I_1\left(N+\frac{2}{3}M\right)}{N}\ln|\tau-\tau_\phi|. \tag{42}$$

Here ϕ_0 and τ_ϕ are constants of integration and $\rho+a=0$ when $\tau=\tau_\phi$. Substituting Eq. (42) into Eq. (24), we find R has a singularity when $\tau=\tau_\phi$:

$$R \sim 2e^{-2\phi_0}|\tau-\tau_\phi|^{I_1[N+(2/3)M]/N+\frac{2I_1\left(N+\frac{2}{3}M\right)}{N}} \times\left\{\frac{3I_1\left(N+\frac{2}{3}M\right)}{N}+2\right\}\frac{1}{(\tau-\tau_\phi)^2}. \tag{43}$$

If $I_1>0$, the second term diverges when $\tau\sim\tau_\phi$. On the other hand, if $I_1<0$, the first term diverges. Therefore there is a singularity if $I\neq 0$.

When $I_1=0$, α also vanishes. When $\rho+a\sim 0$, the behavior of ρ is given by Eq. (40) by putting $\alpha=I_1=0$. Then the behavior of ϕ is given by using Eq. (34),

$$\phi \sim -\left(N+\frac{2}{3}\right)\ln|\tau-\tau_\phi|+\phi_0. \tag{44}$$

Here ϕ_0 is a constant of the integration and $\tau_\phi=\tau_\alpha$. From Eq. (24), we find that there is a curvature singularity when $\tau=\tau_\phi$:

$$R \sim 2e^{2\phi_0}|\tau-\tau_\phi|^{-2[N+(2/3)M]}. \tag{45}$$

Equation (35) can be compared with the system of one particle with unit mass and in the potential $V(\rho)$ in the classical mechanical system when the total energy vanishes. Since the ''kinetic energy'' $\frac{1}{2}(\rho')^2$ is positive, ρ can have its value in the region where $V(\rho)$ is negative. Therefore the following cases can be allowed.

(1) $0>\beta>\alpha$ $(I_2>0)$. In this case, the region with $\rho<\alpha$ and the region with $\rho>\beta$ are allowed. The region $\rho>\beta$ would correspond to an expanding universe but there is always the curvature singularity of Eq. (37) at $\rho+a=\beta$ $(\tau=\tau_\beta)$.

(2) $\alpha=\beta$ $(I_2>0)$. In this case, from the solution (38), we find that there is a singularity when $\tau=\tau_0$ coming from Eq. (38). In case of the expanding universe [$+$ sign in Eq. (38)], Eq. (34) tells $\phi'=0$; i.e., ϕ is a constant as in Nariai space [18]. On the other hand, in the case of the shrinking universe [$-$ sign in Eq. (38)], from Eq. (39), we find that there is a curvature singularity when $\tau-\tau_0=[-(N+M)/12I_2]e^{-a}$ besides $\tau=\tau_0$.

(3) $0>\alpha>\beta$ $(I_2>0)$. The region $\rho+a<\beta$ and the region $\rho+a>\alpha$ are allowed. In the latter case, the shrinking universe turns to expand at $\rho+a=\alpha$ $(\tau=\tau_\alpha)$ but there is always a curvature singularity coming from the singularity as explained around Eq. (41) at $\rho+a=0$ $(\tau=\tau_\phi)$ when the the universe is shrinking $(\tau_\phi<\tau_\alpha)$ if $I_1>0$ or when the universe is expanding $(\tau_\phi>\tau_\alpha)$ if $I_1<0$.

(4) $0=\alpha>\beta$ $(I_1=0)$. When $I_2<0$, the region $\beta<\rho+a<0$ can be allowed. On the other hand, when $I_2>0$, the region $\rho+a<\beta$ and the region $\rho+a>\alpha$ are allowed. In the case of $\rho+a>\alpha$, however, we find from Eq. (44) that there is a curvature singularity when $\rho+a\sim 0$ $(\tau\sim\tau_\alpha=\tau_\phi)$.

(5) $\alpha>0>\beta$ $(I_2<0)$. Only the region $\beta<\rho<\alpha$ can be allowed. There is no solution describing the expanding universe in this case.

As follows from the above analysis in the purely induced gravity case when an expanding universe is constructed due to matter quantum effects one always gets a curvature singularity such as in the case discussed in Ref. [12]. Nevertheless, it is remarkable that the equations of motion in this case admit analytical solutions.

In summary, using an s-wave and large-N approximation we studied gravitational equations of motion with quantum corrections. The analytical solutions representing (non)singular KS cosmology are found. In this derivation, for a nonsingular KS universe the analogy with the Nariai BH (after interchange of time and radial coordinates) is used. In the same way, starting from more complicated multiple horizon BHs with constant curvature one can find other families of nonsingular quantum cosmologies.

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APPENDIX: ANOMALY-INDUCED EFFECTIVE ACTION FOR DILATON-COUPLED SPINORS

In this appendix, we present the conformal anomaly for 2D dilaton coupled spinors (it was derived in [10]). We start from the 2D dilaton-coupled spinor Lagrangian

$$L=\sqrt{-g}f(\phi)\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi, \tag{A1}$$

where ψ is a 2D Majorana spinor, $f(\phi)$ is an arbitrary function, and ϕ is dilaton.

Let us make now the following classical transformation of the background field $g_{\mu\nu}$:

$$g_{\mu\nu}\to f^{-2}(\phi)\tilde{g}_{\mu\nu}. \tag{A2}$$

Then it is easy to see that $\gamma^{\mu}(x)\to f(\phi)\tilde{\gamma}^{\mu}(x)$ and in terms of a new classical metric we obtain the usual Lagrangian noncoupled with a dilaton (minimal) for 2D spinors:

$$L=\sqrt{-\tilde{g}}\bar{\psi}\tilde{\gamma}^{\mu}\partial_{\mu}\psi. \tag{A3}$$

Then we get the following conformal anomaly for dilaton-coupled Majorana spinors, Eq. (A1):

$$\sqrt{-g}T=\sqrt{\frac{-g}{24\pi}}\left[\frac{1}{2}R-\Delta\ln f\right] =\sqrt{\frac{-g}{24\pi}}\left[\frac{1}{2}R-\frac{f'}{f}\Delta f-\frac{(f''f-f'^2)}{f^2}g^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi\right]. \tag{A4}$$

Integrating the anomaly, we find the anomaly-induced effective action W in the covariant, nonlocal form [10]:

$$W=-\frac{1}{2}\int d^2x\sqrt{-g}\left\{\frac{1}{96\pi}R\frac{1}{\Delta}R-\frac{1}{24\pi}R\ln f(\phi)\right\}. \tag{A5}$$

That gives an anomaly-induced effective action for dilaton coupled spinors. It is interesting that adding this W to classical part of the Callan-Giddings-Harvey-Strominger (CGHS) model [20] we get the Russo-Susskind-Thorlacius (RST) model [21] where the last term in W has been introduced by hands in Ref. [21]. Dilaton-coupled quantum spinors give the natural realization of the RST model.

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